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NON-SINGULAR REPRESENTATIONS OF THE GRAVITATIONAL POTENTIAL

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NON-SINGULAR REPRESENTATIONS OF THE GRAVITATIONAL POTENTIAL

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REPORT

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NON-SINGULAR REPRESENTATIONS OF THE GRAVITATIONAL POTENTIAL

Dedicated to:

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NON–SINGULAR REPRESENTATIONS OF THE GRAVITATIONAL POTENTIAL

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Pines' and Gottlieb's Formulations for the gravitational potential provides expressions for the gravitational potential, U, and its derivatives in a co-ordinate system that produces non-singular values. This report summarizes the origin of the singularities due to the spherical co-ordinate system and a discussion of the methods by which the singularity produced by the conventional representation of the gravitational potential is removed by the implementations described in this report.

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1. Introduction

In order to understand the need for alternate formulations, it is important to identify the source of the singularities in the conventional representation of the gravitational potential. There are two singularities that appear within the gravitational potential, the first is where the radius, r, equals zero. However, this is irrelevant for points outside the body (such as an orbiting satellite) since r must be greater than the body radius, a, which is never zero and therefore, r, is never zero. The second singularity, which this report will focus on, is related to the fact that the use of spherical co-ordinates , including latitude and longitude, result in a singularity at the poles when the latitude, ϕ, is equal to $\pm\frac{\pi}{2}$ since longitude is not defined at these points.

This report summarizes the origin of the singularities in the conventional representation and a discussion of the methods through which the singularity in the conventional representation is removed by the implementations described in this report. The existence of these singularities imposes limitations on their applications associated with the orbit characteristics, particularly the orbit inclination.

In the conventional representation, the position vector for a satellite in the orbit of said non-spherical body, however, is usually expressed in an orthogonal, cartesian reference frame, e.g., $\mathbf{R}(x, y, z)$. However, the gravitational potential, U, for a non-spherical body, is usually expressed in spherical co-ordinates, r, the radius of the orbit, ϕ, the geocentric latitude and λ, the longitude angle [2]. Therefore, since r is given by

(1) $$r = (x^2 + y^2 + z^2)^{0.5}$$

where, (x, y, z) are components of the position vector, $\mathbf{R}$, as

(2) $$\mathbf{R} = \left\{ \begin{array}{c} x \\ y \\ z \end{array} \right\} = \left\{ \begin{array}{c} r\cos\phi\cos\lambda \\ r\cos\phi\sin\lambda \\ r\sin\phi \end{array} \right\}$$

And therefore, the conventional representation of the gravitational potential expressed in spherical co-ordinates can be shown to be

(3) $$U = \frac{\mu}{r}\{1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(\frac{a}{r})^n P_{n,m}(\sin\phi)(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda)\}$$

where in this report, a is defined as the body radius and μ is the gravitational body constant (GM) . Furthermore, $P_{n,m}(sin\phi)$ is the Legendre associated function of degree, n, and order, m and where $sin\phi$ is the argument of the Legendre associated function and $C_{n,m}$ and $S_{n,m}$ are the spherical harmonic coefficients.

The Legendre associated function is defined as

(4) $$P_{n,m}(\sin\phi) = (1 - \sin\phi^2)^{m/2}(\frac{1}{2^n n!})(\frac{d^{n+m}}{d\sin\phi^{n+m}}(\sin\phi^2 - 1)^n)$$

It is also important to note the many recursive properties of the Legendre associated function, as discussed by Lundberg and Schutz [5]. For example,

$$P_{n,m}(\sin\phi) = \frac{[(2n-1)\sin\phi P_{n-1,m}(\sin\phi) - (n+m-1)P_{n-2,m}(\sin\phi)]}{n-m} \tag{5}$$

The acceleration term, $\ddot{\mathbf{r}}$, that results from the gravitational force can be generated from the gradient of the gravitational potential, which can be represented by the partial derivatives of U with respect to the three spherical components, which produces the following vector representation

$$\ddot{\mathbf{r}} = \nabla U = \frac{\partial U}{\partial r}\mathbf{u_r} + \frac{1}{r}\frac{\partial U}{\partial \phi}\mathbf{u}_\phi + \frac{1}{r\cos\phi}\frac{\partial U}{\partial \lambda}\mathbf{u}_\lambda \tag{6}$$

where the unit vectors in the spherical co-ordinate system are described by the co-ordinate transformation

$$\left\{\begin{array}{c} \mathbf{u_r} \\ \mathbf{u}_\phi \\ \mathbf{u}_\lambda \end{array}\right\} = \left[\begin{array}{ccc} \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \\ -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \end{array}\right]\left\{\begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array}\right\} \tag{7}$$

However, spherical co-ordinates come with the significant disadvantage, which has been noted previously, that at the poles ($\phi=\pm\frac{\pi}{2}$) the longitude becomes undefined, as at these two poles a point can take on infinite possible longitudes.

By observing this relationship, you can see that the gradient of the gravitational potential is singular at the poles, as in Eq.(6) the term $\frac{1}{cos\phi}$ is undefined as the $\cos\phi$ approaches zero when ϕ approaches $\pm\frac{\pi}{2}$. This is important, as in numerical integration of the ordinary differential equations representing satellite motion, the gravitation force represented by ∇U will be evaluated at different points in the orbit and will likely be evaluated at or near ϕ=$\pm\frac{\pi}{2}$ for a polar orbit. This report summarizes two papers that remove this singularity and result in non-singular representations of the gravitational potential [1], [4].

2. Non-Singular Gravitational Potential: Pines

For Pines' representation, the variables are defined as r, the scale magnitude of the vector $R(x, y, z)$, and the three components of the unit vector, $\hat{R}$, are defined as

$$r = (x^2 + y^2 + z^2)^{0.5} \tag{8}$$

$$\xi = \frac{x}{r} \tag{9}$$

$$\eta = \frac{y}{r} \tag{10}$$

$$\zeta = \frac{z}{r} \tag{11}$$

Therefore

$$\xi^2 + \eta^2 + \zeta^2 = 1 \tag{12}$$

It is important to note here the difference between the co-ordinate system used by Pines, (s, t, u), and that used in this report, (ξ, η, ζ). Pines' co-ordinate system maintains the orthogonality of the conventional co-ordinate system. Using this knowledge, the Pines' formulation replaces $\sin\phi$ in the Legendre associated function with ζ, which produces the Pines' derived Legendre polynomial, $A_{n,m}(\zeta)$

$$A_{n,m}(\zeta) = (\frac{1}{2^n n!})\frac{d^{n+m}}{d\zeta^{n+m}}(\zeta^2 - 1)^n \tag{13}$$

In addition, $\sin m\lambda$ and $\cos m\lambda$ need to replaced with their own non-singular equivalents, defined as

$$r_m(\xi, \eta) = \cos m\lambda \cos m\phi \tag{14}$$

$$i_m(\xi, \eta) = \sin m\lambda \cos m\phi \tag{15}$$

$r_m(\xi, \eta)$ and $i_m(\xi, \eta)$ become the real and imaginary parts of a complex number, $(\xi + i\eta)^m$, where $i = \sqrt{-1}$.

Therefore, with these variables, Pines' representation allows for a new representation of the gravitational potential

$$U = \frac{\mu}{r}\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}(\frac{a}{r})^n A_{n,m}(\zeta)(C_{n,m}r_m(\xi, \eta) + S_{n,m}i_m(\xi, \eta))\} \tag{16}$$

In addition, Pines' representation maintains the recursive nature of the Legendre polynomials through the derived Legendre polynomials, $A_{n,m}(\zeta)$, with the argument, ζ. If

$$A_{n,0}(\zeta) = P_n(\zeta) = \frac{1}{2^n n!}\frac{d^n}{d\zeta^n}(\zeta^2 - 1)^n \tag{17}$$

then

$$A_{n,m}(\zeta) = \frac{d^m}{d\zeta^m}P_n(\zeta) = \frac{1}{2^n n!}\frac{d^{n+m}}{d\zeta^{n+m}}(\zeta^2 - 1)^n \tag{18}$$

As we know, the Legendre polynomial allows for recursion

$$nP_n(\zeta) = \zeta(\frac{d}{d\zeta})P_n(\zeta) - (\frac{d}{d\zeta})P_{n-1}(\zeta) \tag{19}$$

The derived Legendre polynomial also allows for an equivalent recursion equation

$$A_{n,m}(\zeta) = \frac{\zeta A_{n,m+1}(\zeta) - A_{n-1,m+1}(\zeta)}{(n-m)} \tag{20}$$

This allows one to calculate $A_{n,m}(\zeta)$, by reducing any polynomial with an order, m, is less than degree, n + *1*, using the following relationships

$$A_{n,n}(\zeta) = 1 \cdot 3 \cdot 5 \cdots (2n-1) \tag{21}$$

$$A_{n,n-1}(\zeta) = \zeta A_{n,n}(\zeta) \tag{22}$$

Therefore, this maintains the recursive properties of the conventional representation.

3. Derivatives of Non-Singular Gravitational Potential: Pines

Once the gravitational potential is represented in Pines' co-ordinate system, $U(\xi, \eta, \zeta, r)$, it can be shown that Pines' representation has produced a non-singular gravitational potential and non-singular first derivatives. The gravitational force (called the *acceleration force* in Pines' article) was represented conventionally using the spherical unit vectors, $\mathbf{u_r}$, $\mathbf{u}_\phi$ and $\mathbf{u}_\lambda$. However, the force is represented by Pines as scalar coefficients of four vectors: $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and $\hat{R}$. Beginning with the Pines' representation of the gravitational potential and the derived Legendre polynomial

$$U = \frac{\mu}{r}\{1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(\frac{a}{r})^n A_{n,m}(\zeta)(C_{n,m}r_m(\xi,\eta) + S_{n,m}i_m(\xi,\eta))\} \tag{23}$$

$$A_{n,m}(\zeta) = (\frac{1}{2^n n!})\frac{d^{n+m}}{d\zeta^{n+m}}(\zeta^2 - 1)^n \tag{24}$$

Partials derivatives for ∇U can be found for Pines' representation with respect to the vector, R, the cartesian position vector, in order for the gravitational force to be represented in the above vector-basis.

$$\frac{\partial r}{\partial \mathbf{R}} = \hat{R} \tag{25}$$

$$\frac{\partial \xi}{\partial \mathbf{R}} = \frac{1}{r}\hat{\imath} - \frac{\xi}{r}\hat{R} \tag{26}$$

$$\frac{\partial \eta}{\partial \mathbf{R}} = \frac{1}{r}\hat{i} - \frac{\eta}{r}\hat{R} \tag{27}$$

$$\frac{\partial \zeta}{\partial \mathbf{R}} = \frac{1}{r}\hat{i} - \frac{\zeta}{r}\hat{R} \tag{28}$$

Therefore, Pines' representation is a non-singular representation of the gradient of the gravitational porential

$$\ddot{\mathbf{r}} = \nabla U = (\frac{\partial U}{\partial r} - \frac{\xi}{r}\frac{\partial U}{\partial \xi} - \frac{\eta}{r}\frac{\partial U}{\partial \eta} - \frac{\zeta}{r}\frac{\partial U}{\partial \zeta})\hat{R} + \frac{1}{r}\frac{\partial U}{\partial \xi}\hat{i} + \frac{1}{r}\frac{\partial U}{\partial \eta}\hat{j} + \frac{1}{r}\frac{\partial U}{\partial \zeta}\hat{k} \tag{29}$$

With this representation of the gradient of the gravitational potential, there are four first partials that need to be evaluated and proven to be non-singular: $\frac{\partial U}{\partial r}$, $\frac{\partial U}{\partial \xi}$, $\frac{\partial U}{\partial \eta}$ and $\frac{\partial U}{\partial \zeta}$.

For comparison, Denkin provides a representation of the three first derivatives of gravitational potential in the conventional spherical co-ordinate system [3].

$$\frac{\partial U}{\partial r} = -\frac{\mu}{r^2}\{\sum_{n=1}^{\infty}\sum_{m=1}^{n}(n+1)(\frac{a}{r})^n P_{n,m}(\sin\phi)(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda)\} \tag{30}$$

$$\frac{\partial U}{\partial \lambda} = \frac{\mu}{r}\{\sum_{n=1}^{\infty}\sum_{m=1}^{n} m(\frac{a}{r})^n P_{n,m}(\sin\phi)(S_{n,m}\cos m\lambda - C_{n,m}\sin m\lambda)\} \tag{31}$$

With the remaining derivative being dependent on the derivative of the summation of the Legendre polynomials and, therefore, singular due to the element of $\frac{1}{cos\phi}$ that appears. Pines' representation of the partial derivatives utilizes the recursive relationships between the components of $(\xi + i\eta)^m$. These relationships come from the properties of complex differentiation

$$\frac{\partial r_m}{\partial \xi} = \frac{\partial i_m}{\partial \eta} = m r_{m-1}(\xi, \eta) \tag{32}$$

$$-\frac{\partial r_m}{\partial \eta} = \frac{\partial i_m}{\partial \xi} = m i_{m-1}(\xi, \eta) \tag{33}$$

These identities lead to the following relationship from complex variable algebra

$$r_m(\xi, \eta) = \xi r_{m-1}(\xi, \eta) - \eta i_{m-1}(\xi, \eta) \tag{34}$$

$$i_m(\xi, \eta) = \xi i_{m-1}(\xi, \eta) + \eta r_{m-1}(\xi, \eta) \tag{35}$$

And therefore

$$\xi(\frac{\partial r_m}{\partial \xi}) - \eta(\frac{\partial i_m}{\partial \eta}) = r_m \tag{36}$$

$$\xi(\frac{\partial i_m}{\partial \xi}) + \eta(\frac{\partial r_m}{\partial \eta}) = i_m \tag{37}$$

Pines' representation also includes the following substitution for the partial derivatives:

$$\rho_{n+1} = (\frac{a}{r})^{n+1} \tag{38}$$

$$\rho_n = (\frac{a}{r})^n \frac{\mu}{r} \tag{39}$$

In addition, the Pines' representation encourages the use of a new set of coefficients that are combinations of $C_{m,n}$ and $S_{m,n}$, defined as

$$Z_{n,m}(\xi,\eta) = C_{n,m} r_m(\xi,\eta) + S_{n,m} i_m(\xi,\eta) \tag{40}$$

$$\Xi_{n,m}(\xi,\eta) = C_{n,m} r_{m-1}(\xi,\eta) + S_{n,m} i_{m-1}(\xi,\eta) \tag{41}$$

$$H_{n,m}(\xi,\eta) = S_{n,m} r_{m-1}(\xi,\eta) - C_{n,m} i_{m-1}(\xi,\eta) \tag{42}$$

With the newly defined coefficients, the gravitational potential and the three partial derivatives with respect to ξ, η and ζ can be defined as scalar coefficients a_1, a_2 and a_3.

$$U = \sum_{n=0}^{\infty} \rho_n \sum_{m=0}^{n} A_{n,m}(\zeta) Z_{n,m}(\xi, \eta) \tag{43}$$

$$a_1 = \frac{1}{r}\frac{\partial U}{\partial \xi} = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m}(\zeta) m \Xi_{n,m} \tag{44}$$

$$a_2 = \frac{1}{r}\frac{\partial U}{\partial \eta} = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m}(\zeta) m H_{n,m} \tag{45}$$

$$a_3 = \frac{1}{r}\frac{\partial U}{\partial \zeta} = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m+1}(\zeta) Z_{n,m} \tag{46}$$

a_1, a_2 and a_3 are the scalar coefficients of the vector basis, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. Lastly, the coefficient for the radial vector, $\hat{R}$, is defined using the recursion formula for the derived Legendre polynomial, and leads to

$$a_4 = \frac{\partial U}{\partial R} - \frac{\xi}{r}\frac{\partial U}{\partial \xi} - \frac{\eta}{r}\frac{\partial U}{\partial \eta} - \frac{\zeta}{r}\frac{\partial U}{\partial \zeta} = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n+1,m+1}(\zeta) Z_{n,m} \tag{47}$$

Therefore, the gravitational force can be presented in a more straightforward to code form of

$$\ddot{\mathbf{r}} = \nabla U = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} + a_4\hat{R} \tag{48}$$

In conclusion, Pines' representation gives a set of non-singular equations for the gravitational potential, the gradient of the gravitational potential and the first derivatives of the gravitational potential .

4. Non-Singular Gravitational Potential: Gottlieb

Gottlieb's Representation is an additional alternative representation of the gravitational potential [4]. The three variables defined by Gottlieb include the scalar magnitude of the position vector, r, and two additional variables, defined as

$$\zeta = \frac{z}{r} \tag{49}$$

$$\beta_{n,m} = C_{n,m} C_m + S_{n,m} S_m \tag{50}$$

where $C_{n,m}$ and $S_{n,m}$ have been defined previously and a different set of coefficients, C_m and S_m, are defined as

$$C_m = \rho^m \cos m\lambda \tag{51}$$

$$S_m = \rho^m \sin m\lambda \tag{52}$$

Where λ is the longitude defined in the first section and ρ is defined as

$$\rho^2 = r^2 - \zeta^2 \tag{53}$$

Using the above definition for ζ, Gottlieb redefines the argument of the Legendre polynomial as 1 - ζ^2. He defines the Legendre-like polynomial as follows:

$$P_{n,m}(1-\zeta^2) = (1-\zeta^2)^{m/2}\frac{\partial^m P_n}{\partial \zeta^m} = \frac{\rho^m}{r^m}P_n^m \tag{54}$$

Where P_n^m is defined as

$$P_n^m = \frac{\partial^m P_n}{\partial \zeta^m} \tag{55}$$

These variables produce a gravitational potential defined as

$$U = \frac{\mu}{r}\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}(\frac{a}{r})^n \beta_{n,m}(\frac{P_n^m}{r^m})\} \tag{56}$$

In addition, it is important to note the maintenance of the recursive relationship of the Legendre Polynomial through the Derived Legendre Function, P_n^m

$$P_n^m = \frac{((2n-1)(\zeta)P_{n-1}^m - (n+m-1)P_{n-2}^m)}{n-m} \tag{57}$$

5. Derivatives of Non-Singular Gravitational Potential: Gottlieb

Gottlieb's co-ordinate system allows for the gravitational force to be presented as follows

$$\ddot{\mathbf{r}} = \nabla U = \frac{\partial U}{\partial r}\frac{\partial r}{\partial \mathbf{R}} + \frac{\partial U}{\partial \zeta}\frac{\partial \zeta}{\partial \mathbf{R}} + \frac{\partial U}{\partial \beta_{n,m}}\frac{\partial \beta_{n,m}}{\partial \mathbf{R}} \tag{58}$$

where

$$\frac{\partial U}{\partial r} = -\frac{\mu}{r^2}\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}(\frac{a}{r})^n \beta_{n,m}(\frac{(n+m+1)P_n^m}{r^m})\} \tag{59}$$

$$\frac{\partial U}{\partial \zeta} = \frac{\mu}{r}\{\sum_{n=2}^{\infty}\sum_{m=0}^{n}(\frac{a}{r})^n \beta_{n,m}(\frac{P_n^{m+1}}{r^m})\} \tag{60}$$

$$\frac{\partial U}{\partial \beta_{n,m}} = \frac{\mu}{r}\{\sum_{n=2}^{\infty}\sum_{m=0}^{n}(\frac{a}{r})^n(\frac{P_n^m}{r^m})\} \tag{61}$$

It is also important to note the partial derivatives of the three variables with respect to the radial vector, $\mathbf{R}(x, y, z)$

$$\frac{\partial r}{\partial \mathbf{R}} = \hat{R} \tag{62}$$

(63) $$\frac{\partial \zeta}{\partial \mathbf{R}} = \frac{1}{r}\hat{k} - \frac{\zeta}{r}\hat{R}$$

In order to derive the partial of $\beta_{n,m}$ with respect to $\mathbf{R}$, you need the following two partials

(64) $$\frac{\partial \rho}{\partial \mathbf{R}} = \frac{x}{\rho}\hat{\imath} + \frac{y}{\rho}\hat{\jmath}$$

(65) $$\frac{\partial \lambda}{\partial \mathbf{R}} = -\frac{y}{\rho^2}\hat{\imath} + \frac{x}{\rho^2}\hat{\jmath}$$

Therefore

(66)
$$\frac{\partial \beta_{n,m}}{\partial \mathbf{R}} = mC_{n,m}(\rho^{m-1}\cos m\lambda \frac{\partial \rho}{\partial \mathbf{R}} \rho^m \sin m\lambda \frac{\partial \lambda}{\partial \mathbf{R}}) + mS_{n,m}(\rho^{m-1}\sin m\lambda \frac{\partial \rho}{\partial \mathbf{R}} \rho^m \cos m\lambda \frac{\partial \lambda}{\partial \mathbf{R}})$$

Using above definitions, this can be re-written as

(67) $$\frac{\partial \beta_{n,m}}{\partial \mathbf{R}} = m(C_{n,m}C_{m-1} + S_{n,m}S_{m-1})\hat{\imath} - m(C_{n,m}S_{m-1} - S_{n,m}C_{m-1})\hat{\jmath}$$

Gottlieb introduces a series of coefficients, H, J, K and Γ in order to allow for a less complex representation of the gradient of gravitational potential

$$J_n = \sum_{m=1}^{n} m \frac{P_n^m}{r^{m-1}} (C_{n,m} C_{m-1} + S_{n,m} S_{m-1}) \tag{68}$$

$$K_n = -\sum_{m=1}^{n} m \frac{P_n^m}{r^{m-1}} (C_{n,m} S_{m-1} - S_{n,m} C_{m-1}) \tag{69}$$

$$\Gamma_n = C_{n,0}(n+1) P_n^0 + \sum_{m=1}^{n} (1+n+m) \frac{P_n^m}{r^m} (C_{n,m} C_m + S_{n,m} S_m) \tag{70}$$

$$H_n = C_{n,0} P_n^1 + \sum_{m=1}^{n} \frac{P_n^{m+1}}{r^m} (C_{n,m} C_m + S_{n,m} S_m) \tag{71}$$

$$J = -\frac{\mu}{r^2} \sum_{n=1}^{\infty} (\frac{a}{r})^n J_n \tag{72}$$

$$K = -\frac{\mu}{r^2} \sum_{n=1}^{\infty} (\frac{a}{r})^n K_n \tag{73}$$

$$H = -\frac{\mu}{r^2} \sum_{n=1}^{\infty} (\frac{a}{r})^n H_n \tag{74}$$

$$\Gamma = -\frac{\mu}{r^2} (1 + \sum_{n=1}^{\infty} (\frac{a}{r})^n \Gamma_n) \tag{75}$$

$$\Lambda = \Gamma + \zeta H \tag{76}$$

Therefore, the gravitational force can be written simply as

$$\ddot{\mathbf{r}} = \nabla U = \Lambda \hat{R} - J\hat{i} - K\hat{j} - H\hat{k} \tag{77}$$

6. Summary

In summary, the gravitational potential can be represented in the conventional way, in spherical co-ordinates (r, ϕ, λ). However, these co-ordinates produce an obvious singularity at $\pm\frac{\pi}{2}$ in the gradient of the function, U, due to the longitude having becoming undefined at the pole latitudes. Therefore, this paper has described two equivalent alternative representations of the gravitational potential.

Pines' representation describes the gravitational potential with four variables (r, ξ, η, ζ) that remove the singularity present at polar latitudes,

$$U = \frac{\mu}{r}\{1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(\frac{a}{r})^n A_{n,m}(\zeta)(C_{n,m}r_m(\xi,\eta) + S_{n,m}i_m(\xi,\eta))\} \tag{78}$$

As well as a non-singular gravitational force

$$\ddot{\mathbf{r}} = \nabla U = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} + a_4\hat{R} \tag{79}$$

Gottlieb's representation, though not significantly different from Pines', is described by three variables $(r, \zeta, \beta_{n,m})$, which also removes the polar singularity with the gravitational potential

$$U = \frac{\mu}{r}\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}(\frac{a}{r})^n \beta_{n,m}(\frac{P_n^m}{r^m})\} \tag{80}$$

As well as a non-singular gravitational force

$$\ddot{\mathbf{r}} = \nabla U = \Lambda \hat{R} - J\hat{i} - K\hat{j} - H\hat{k} \tag{81}$$

Both approaches remove the polar singularities, however, the two different representations are not significantly different and produce similar outcomes with similar levels of complexity.

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